Rentian scaling for the measurement of optimal embedding of complex networks into physical space

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The London Underground is one of the largest, oldest and most widely used systems of public transit in the world. Transportation in London is constantly challenged to expand and adapt its system to meet the changing requirements of London’s populace while maintaining a cost-effective and efficient network. Previous studies have described this system using concepts from graph theory, reporting network diagnostics and core–periphery architecture. These studies provide information about the basic structure and efficiency of this network; however, the question of system optimization in the context of evolving demands is seldom investigated. In this paper we examined the cost effectiveness of the topological–physical embedding of the Tube using estimations of the topological dimension, wiring length and Rentian scaling, an isometric scaling relationship between the number of elements and connections in a system. We measured these properties in both two- and three-dimensional embeddings of the networks into Euclidean space, as well as between two time points, and across the densely interconnected core and sparsely interconnected periphery. While the two- and three-dimensional representations of the present-day Tube exhibit Rentian scaling relationships between nodes and edges of the system, the overall network is approximately cost-efficiently embedded into its physical environment in two dimensions, but not in three. We further investigated a notable disparity in the topology of the network’s local core versus its more extended periphery, suggesting an underlying relationship between meso-scale structure and physical embedding. The collective findings from this study, including changes in Rentian scaling over time, provide evidence for differential embedding efficiency in planned versus self-organized networks. These findings suggest that concepts of optimal physical embedding can be applied more broadly to other physical systems whose links are embedded in a well-defined space, and whose topology is constrained by a cost function that minimizes link lengths within that space.

Keywords: Spatial embedding; Topophysical networks; Rentian scaling; Core-periphery structure; Transportation.

1. Introduction

In recent years, the field of network science made important advances in the tools available for understanding the topological organization of complex systems. Such tools can be used to quantify network
organization by measuring topological statistics, or network diagnostics, of a system, and can thereby offer insights into system function. One particularly useful diagnostic is known as network efficiency, which indirectly measures the ease of information or product transfer through a network, under the assumption that information is routed along the shortest possible topological paths [1]. The concept of network efficiency is thought to play a role in many different complex systems, such as networks of neurons, systems of roads or groups of cellular interactions. In each of these systems, the goal is to perform a certain function while expending minimal energy. In this context, network efficiency is inherently related to network optimization: how well is a given system optimized for a particular task or outcome?

How does one define network optimization? To state that a network is optimized requires an understanding of how efficiently the system is organized in light of its constraints. A common constraint on network structure is cost-efficiency: the ability to enable maximal efficiency for minimal cost. The basis of network optimization involves analysis of the costs of the network architecture versus the benefits of that network architecture to the system. Although many networks are studied as abstract topological structures, a growing number of studies integrate information about the physical space in which the system lives. For example, physical embedding is integral to the study of spatial networks [2], which are observed in many common biological, technological and social systems. With increasing interest in the study of such systems, come a critical need to understand whether a network is optimally embedded into its physical environment. Consider very-large-scale integrated (VLSI) circuits, which are codified in terms of the physical length of the wires that constitute them [3]. Designing an optimal VLSI system requires one to minimize the cost (i.e. length of wire) of embedding that system onto a physical, two-dimensional circuit board. Similarly, neurobiological networks are physically embedded in neural tissue, and mounting evidence suggests that such networks display near-optimal physical embedding as illustrated by the nearly minimal physical length of connections [4]. Underlying metabolic costs of cellular growth and the organization of connections that allow more efficient communication between different regions in the brain are hypothesized to drive this wiring minimization [5].

One analytical method that offers insight into the optimality of a system’s physical embedding is the determination of Rentian scaling, a salient aspect of fractal network design that measures the hierarchical modularity of a system through calculation of the relationship between the number of external signal connections to a logic block (i.e. the number of ‘pins’) and the number of logic gates in the logic block [6]. Electrical engineers originally developed Rent’s rule—in which these two variables scale with one another in log–log space—to describe integrated circuit design with optimization measures based on analysis of costs versus benefits to the system. However, this rule may be applied to any biological, technological or social system in which there is an interest in minimizing a cost while also allowing for the development of specified topological patterns. In such a physically embedded system, the optimal network configuration likely displays a balance between wiring minimization and the value of the system output. For example, in transportation networks it is costly to build multiple routes to the same destination; however, doing so is valuable to the people using the transit system as it greatly reduces the number of transfers required to reach their destination. These concepts of Rentian scaling and network optimization previously demonstrated that brain circuits are economically embedded in a manner similar to VLSI circuits [5]. It is intuitively plausible, though completely untested, that the London Underground (the Tube) exhibits a similar optimally embedded architecture.

The Tube is the primary subway transportation network for the Greater London area in the UK. Construction of the Tube started in 1863 and the system was the world’s first underground railway. Today, the Tube comprised 270 stations and 402 km of track and serves 1.23 billion passengers annually (https://www.tfl.gov.uk/). By treating the Tube as a network in which stations are represented by nodes and tracks between stations are represented by edges, previous studies used tools from graph theory to better
understand the structure and optimal function of the system [7–11]. Several studies also evaluated the efficiency of movement through the Tube network, including the cost of transferring within the network [7, 8, 12]. Guo and Wilson found that transferring significantly reduces efficiency of movement through the system [8]. Additionally, other studies assessed network vulnerability, specifically in reference to system congestion [13–16], contagion spreading [17], critical lines [18, 19] and the terrorist attacks in 2005 [9, 20]. Out of nearly 3 million combinations of possible station attacks, targeted attack produced near optimal disruption to the system. Finally, Rombach et al. [10] investigated the modularity of the system and established the existence of core–periphery structure within the Tube.

In the majority of these studies, the system is treated as an abstract entity. Although transfers, disruptions and basic network diagnostics are examined in a number of studies, very little work examines the physical embedding of the Tube. In 2004, Csányi and Szendrói [11] analysed the fractal scaling of networks embedded within geographical constraints. In these networks, neighbourhoods of nodes grow according to a power law; results of this study found the Tube to exhibit fractal scaling. In a paper published before Csányi and Szendrói, Kim et al. [21] analysed the transportation system of Seoul, South Korea over a series of several years and demonstrated an increase in the fractal dimension over time. In addition to studies that directly examine the Tube, several studies address the concept of network geometry in other transport systems, beginning with Gastner and Newman in 2006 [25], and also covering core–periphery structure [22], efficiency of star-like graphs [23], optimal navigation of lattice structures with long-range connections [24] and strategies for maintaining robustness as a network grows [26]. These studies collectively address the question of topology-based fractal scaling and geometry-based notions of optimality but they do not address the embedding of the network into physical space. In this study, we examined the organization of the Tube in relation to its physical embedding.

Similar to VLSI circuits [3], it is important for complex transportation networks to be cost-efficiently embedded into the physical space in which they exist to minimize the cost of track and create an efficient system for travellers. Utilizing methods previously developed for analysing VLSI and neurobiological systems, we examine Rentian scaling, the topological fractal dimension and wiring length to better understand if the Tube is cost-efficiently embedded into its physical environment [5]. This study begins with a topological analysis of the Tube network and comparison to benchmark networks to develop a picture of network organization. Subsequently, we consider the physical location of nodes or stations. Previous work examining the Tube (and other transportation systems), primarily focuses on the topology of the system [27] and does not account for the location of network nodes in physical space (latitude and longitude). In the final section of this paper, the Tube is, for the first time, examined as a three-dimensional network to better understand the efficiency of its physical embedding. Through these analyses, we hypothesize that the two-dimensional physically embedded network will have a lower level of embedding efficiency compared to the three-dimensional physical network, which will provide a more accurate measurement of topological–physical embedding as well as a better representation of the true complexity present in this transportation network.

2. Methods

2.1 Two-dimensional network construction and analysis

We represented the present-day Tube as a network using information provided by Transport for London (TFL), including the latitudinal and longitudinal positions of individual Tube stations and the connections between each station (London Datastore, http://data.london.gov.uk/). We represented train stations as nodes with the tracks connecting stations treated as edges. Using MATLAB (R2015b, The MathWorks,
Inc., Natick, MA, USA), we collated these data in a binary, undirected adjacency matrix consisting of 306 nodes and 353 edges (Fig. 1). We refer to this network as the topological network.

2.2 Three-dimensional network construction

We also represented the Tube as a three-dimensional network to study the physical properties of the system. This physical network describes the location of each station as accurately as possible, incorporating geographical information of latitude, longitude and depth. We obtained these physical locations from the geographical coordinates used in the two-dimensional network and the \( z \)-dimensional elevation above sea level of each station (data provided by Transport for London). The addition of this physical information enabled us to determine if the incorporation of depth information into the Rentian scaling analysis offered novel insights into system architecture or function. Although many Tube stations have different depths for specific tracks, in this analysis the average depth of the station is used. Several stations did not have depth information available; we assigned depths to the stations with unavailable data equal to the average depth of the other stations in the network. The elevations in the Tube network are adjusted by the use of Tunnel datum. Tunnel datum is a conversion used in designing tunnels which pass below sea level. For example, for the Tube, a tunnel datum conversion factor of +100 m is used and therefore a depth of −60 m is 40 m above tunnel datum (ATD), used in this study. A visualization of the resulting three-dimensional network and the distribution of station elevation are presented in Fig. 2. For ease of reference, we refer to this network as the physical network to distinguish it from the topological network defined in the previous section.

2.3 Benchmark network comparison

We compared network diagnostics of the present-day Tube network to two flavours of benchmark random networks: the Erdős–Rényi (ER) random graph model [28, 30] and the random geometric graph [29]. Both benchmark networks contain the same number of nodes and edges as the Tube network; however, the placement of nodes in the ER random graph model is prescribed by the location of nodes in the Tube network and connectivity is bound by a Poisson degree distribution (Fig. 3C). In the geometric graph, nodes are placed uniformly at random within a box prescribed by the size of the Tube network and nodes are connected only to nearby nodes via edges, producing separated clusters of interconnected nodes (Fig. 3D) [29]. In both the ER and geometric models, we performed topological and physical embedding analyses (detailed in the next section) across 100 simulations and compared the results to those observed in the present-day Tube network.

2.4 Topological embedding analysis

Utilizing similar methodology as that outlined in Bassett et al., we calculated the topological Rentian scaling, fractal dimension and the Euclidian dimension [5, 33]. To assess topological Rentian scaling, we calculated the fractal dimension (\( d_B \)) (Fig. 4A) using a box counting algorithm that measures the topological size (\( L \)) and the number of boxes (\( N_B \)) [31, 32]. The topological size \( L \) is the size of boxes that partition the network into halves, quarters, etc. in topological space. The number of boxes \( N_B \) is the count of boxes required to cover all nodes in the network as box size is varied from 1 to \( L_{\text{max}} \). Due to the non-deterministic nature of the algorithm, we performed these calculations multiple times to ensure reliability of results. We then fit a power law function to the \( N_B \) versus \( L \) curve using robust multilinear
Fig. 1. The tube as a network. The design of the network includes information about the locations of Tube stations in London (latitude and longitude coordinates) and the available connections between the various stations (undirected and binary edges). These pieces of information are illustrated in (A) the Tube network map, (B) the Tube lines highlighted on a geographical map of London and (C) the present-day Tube network, combining both connection and geographic information.
Fig. 2. Physical embedding of the tube. The three-dimensional Tube network, as viewed from the southwest corner of the city, and the distribution of station elevation (inset).

regression in Matlab. The exponent of this fit is an estimate of the network’s fractal dimension, $d_B$ [33]:

$$N_B = L^{-d_B}.$$ 

To calculate the topological Rent’s exponent, we used hMETIS software [34] to recursively partition the network and subsequently count the number of interbox edges ($e$) against the number of nodes in each box ($n$), at each level of partitioning. To assess power law scaling, we fit the $n$ versus $e$ data over all partitions using robust multilinear regression, with an iteratively reweighted linear squares calculation and a bisquare weighting function [5, 31]. The topological Rent exponent, $p_T$, is defined as the slope of $n$ versus $e$ in log–log space.

The topological Rent exponent is related to the minimum Rent exponent, $p_{\text{min}}$, for a given set of nodes and edges through the inequality:

$$p_T \geq p_{\text{min}}.$$ 

Previous studies [3] indicate that the fractal dimension, $d_T$ is related to $p_{\text{min}}$ by:

$$p_{\text{min}} = 1 - 1/d_T.$$ 

Given that the topological Rent exponent is greater than the minimum Rent exponent, $p_T$ can be estimated according to the relationship \[ p_T \geq 1 - 1/d_T. \]

Although this relationship exists, all topological exponents presented in this paper are directly calculated using recursive partitioning of the network in hMETIS software.

Finally, the Euclidian dimension, $d_E$, of the network is equal to the physical dimension in which the network is embedded, and we compared this value to the estimated value of the topological dimension, $d_T$. This comparison provided an estimate of the cost of embedding the network, as systems with complex connectivity (large $d_T$) are more challenging to embed into physical space than systems with simple topologies, particularly in the case where $d_T > d_E$.

### 2.5 Assessment of physical embedding

In addition to the topological properties of the network, we also considered the physical properties of the network. We computed the physical Rentian scaling parameters using the Brain Connectivity Toolbox (BCT) [30]. We defined 5,000 physical partitions of the network using the latitude and longitude coordinates of the nodes and the adjacency matrix of the Tube network. Here, a partition is a physical volume—specifically a cube—containing some stations and tracks. For each partition, we calculated the
Fig. 4. Methods to assess characteristics of the topological and physical embedding. (A) We calculated the fractal dimension using a box counting algorithm in which we count the number of boxes \( N \) of topological size \( L \) that are required to cover the network. Note: Though the network is shown here in its physical domain, the box counting algorithm only uses the network’s topology. (B) The physical Rent’s exponent is calculated by placing 5,000 randomly sized boxes uniformly at random on the physically embedded network and (C) calculating the number of nodes \( n \) in each box and the number of edges \( e \) crossing the boundary of each box. We determine whether the network displays physical Rentian scaling by assessing whether \( n \) and \( e \) scale with one another in log–log space.

number of nodes \( (n) \) within the partition and the number of edges \( (e) \) crossing the boundaries of the partition (Figs. 4(B,C)). To avoid edge effects, we only examined partitions with \( n < M/2 \), where \( M \) is the total number of nodes in the network. To assess power-law scaling, we fit the \( n \) versus \( e \) data over all 5,000 partitions using robust multilinear regression, with an iteratively reweighted linear squares calculation and a bisquare weighting function. The physical Rent exponent, \( p \), is defined as the slope of \( n \) versus \( e \) in log–log space. In the context of transportation networks specifically, the subgraphs defined by the bounding boxes provide analysis of transportation flow in and out of particular regions of the network relative to the number of stations in a region, analogous to the flow of electricity in a circuit and the relative number of source/sink nodes.

These methods describe the calculations for the topological dimension the Euclidean dimension, and the mean connection distance between nodes, which we refer to as \( \hat{r} \) [35]. With these variables, it is possible to ask whether the network is cost-effectively embedded into physical space. The average connection length or node-to-node spacing \( (r) \) is the ratio of the linear extent of the system in Euclidian space \( (d_E) \) to that in topological space \( (d_T) \) and in order to lay out a graph of dimension \( d_T \), we use the relationship

\[
\kappa (d_T, d_E) = \hat{r} n^{-1/d_E + 1/d_T}
\]

and conclude that the network is cost-effectively embedded if \( \kappa \sim 1 \) [3].
2.6 Comparison to the early tube network

The early Tube network refers to the network of the London Underground in the year 1900. To construct the early Tube network, we use historical information available from TFL. Stations that did not exist in 1900 were removed from the modern Tube network, and connections that did not yet exist were replaced with a 0. In total, there were 94 nodes and 205 edges in the pre-1900 Tube network (Fig. 3B). We performed the same set of analyses on the topological and physical representations of the early Tube network as those we performed on the present-day Tube network (e.g. benchmark network comparison and embedding analysis).

2.7 Core-periphery architecture and Rentian scaling

To address the variations in network architecture across the Tube system, we examined one aspect of the Tube’s meso-scale structure: stations in the core and periphery of the network (Fig. 1). Core nodes are defined as those that are densely connected to other nodes of the network whereas peripheral nodes are more sparsely connected [10]. We defined core and periphery using k-core decomposition, similarly to the methods of [36], with \( k = 3 \), producing five k-core shells. The four innermost shells define the core \( (N = 47) \) and the outermost shell defines the periphery \( (N = 259) \) (Fig. 5). These definitions are further substantiated by the betweenness centrality metric (Fig. 5, inset), which calculates the fraction of all shortest paths in the network that contain a given node [37]. Nodes with high centrality participate in many of the shortest connections within a network. The sorted plot of centrality exhibits a notable inflection in slope at \( N = 260 \), once again partitioning the network into core and periphery. Subnetwork definition
via the k-core decomposition and betweenness centrality demonstrates an overlap over a majority of the nodes in the core network (30/47 nodes). For the remaining analyses in this paper, the core and periphery are defined using the k-core decomposition technique.

3. Results

3.1 Topological embedding analysis

The fractal dimension is a statistical index of the complexity of a network. We can assess the fractal nature of a network by evaluating the degree of topological invariance over different length-scale transformations. Using the relationship between the number of boxes \(N_B\) used to tile a network and the box size \(L\), we can obtain a naive estimate of the fractal dimension of the Tube network by the slope of a power-law fit to the \(N_B\) versus \(L\) curve (Fig. 6). Such an estimate suggests that the fractal dimension of the network is 1.73. Yet, it is clear from the data that the curve displays two distinct sections that behave quite differently from one another, and therefore a power-law fit to these data does not appropriately represent either portion of the curve.

We therefore examine the two portions of the curve separately (\(L \leq 20\) and \(L > 20\)). At smaller box sizes, we observed that the estimated \(d_B = 1.12\), whereas at larger box sizes, the estimated \(d_B = 4.73\). It is possible that the degree of physical embedding is related to length scale and that the network is approximately one-dimensional at small length scales and more complex over larger distances. Yet, it is important to keep in mind that the data used to estimate the first section of the curve \((d_B = 1.12)\) cover over a decade of the \(x\)-axis values, while the data used to estimate the second section of the curve \((d_B = 4.73)\) cover less than a single decade of the \(x\)-axis values. Thus, statistically our confidence is greater in the estimates from the first section of the curve. With this statistical caveat in mind, we investigate the fractal dimension at both length scales, although with greater scepticism regarding the second section of the curve. In contrast, the ER random graph displays an average \(d_B = 2.82\) and the random geometric model displays an average \(d_B = 0.41\) (Fig. 6(C,D)). Unlike the Tube system’s disparity in topological dimension at varying length-scales, the benchmark networks exhibited greater consistency in topological dimension across the length scales measured in this study.

In addition to computing the fractal dimension, \(d_B\), the topological Rent’s exponent, \(p_T\) is also calculated for each network. We find that \(p_T = 0.31\) for the present-day Tube network, while the ER random graph model displays an average \(p_T = 0.81\) and the random geometric model has a topological Rent’s exponent of \(p_T = 0\).

3.2 Physical embedding analysis

We next turn to an examination of the physical architecture of the Tube network by assessing Rentian scaling properties of the system. Specifically, we partition the network into randomly sized cubes (see Section 2), and study the relationship between the number of nodes in each cube and the number of edges that cross the boundary of the cube (Fig. 7A). We observe that these two variables scale with one another in log–log space. We quantify this scaling relationship by fitting a power law to the data and we estimate the Rent’s exponent to be \(p = 0.4\). In networks that have been optimally embedded, the physical Rent’s exponent is identical to the topological Rent’s exponent: \(p = p_T\). Yet, in the Tube network, we observe that the physical Rent’s exponent is larger than the topological Rent’s exponent: \(p = 0.40 > p_T = 0.31\). These values indicate that the Tube is approximately cost-efficiently embedded.
Fig. 6. Estimating the Topological Fractal Dimension of a Network. Using a box counting method, we assess the relationship between the number of boxes ($N$) and the size of boxes ($L$). (A) Relationship between $N$ and $L$ for the present-day Tube network. We fit the data points with three separate curves: one curve for all data points (grey), one curve for boxes of size $<20$ (pink) and one curve for boxes of size $>20$ (blue). (B) Relationship between $N$ and $L$ for the pre-1900 Tube network. (C) Relationship between $N$ and $L$ for a representative ER random network. (D) Relationship between $N$ and $L$ for a representative random geometric network.
Fig. 7. Evolution of the Tube Network across the Last Century. Physical Rentian scaling of the number of edges (e) to the number of nodes (n) within each of the partitions of the physically embedded network for (A) the present day two-dimensional Tube network, (B) the present day three-dimensional Tube network, (C) the pre-1900 two-dimensional Tube network, (D) the pre-1900 three-dimensional Tube network.
We next ask whether these conclusions are consistent if we examine the three-dimensional Tube network representation as opposed to the two-dimensional Tube network representation. While the topological dimension remains the same in both cases, the physical Rent’s exponent can change due to its dependence on the node’s coordinates. The physical Rent’s exponent of the three-dimensional Tube network was $p = 0.77$ (Fig. 7B). In three-dimensional space, the larger Rent’s exponent indicates increased network complexity and also lack of efficient embedding, as the Rent’s exponent is much greater than the theoretical exponent ($p = 0.77 > p_T = 0.31$).

As an interesting comparison, we assessed the Rentian scaling properties of the benchmark networks, with the node locations of the ER network defined by node locations in the Tube network and node locations in the geometric network defined randomly within the bounds of the Tube network. Over 100 sample networks, we found that the two-dimensional ER random graph model displayed a mean Rent’s exponent of $p = 0.85$ and the two-dimensional geometric random model displayed a mean Rent’s exponent of $p = 0.005$, with 98 out of 100 networks exhibiting $p = 0$. These networks are not efficiently embedded in physical space given that $p > p_T$ for the ER network ($p = 0.85 > p_T = 0.63$) and the majority of random geometric networks lacks Rentian scaling. Similarly, the Rent’s exponent for the three-dimensional ER random graph model was $p = 0.87 > p_T = 0.63$ and the Rent’s exponent for the three-dimensional geometric model was $p = 0$, indicating inefficiency of embedding.

3.3 Comparison of the modern Tube network versus 1900
To understand the evolving complexity of the Tube network over time, we compared the topological and physical properties of the present-day Tube system to that observed in the year 1900. An important feature of this comparison is that in 1900 there were significantly fewer stations and fewer tracks between these stations, which certainly influenced the topological and physical properties of the network. The early Tube network had a smaller core and more tree-like structure, yet did exhibit fractal scaling, with a fractal dimension $d_B = 0.91$ (Fig. 6B). However, this value was much lower than the fractal scaling of the present-day network, which was 1.12 at smaller length scales. Finally, we observed that the early Tube network did not exhibit Rentian scaling either topologically or physically (Fig. 7(C,D)).

3.4 Rentian scaling in core and periphery networks
To address the question of network meso-scale architecture on the topology and physical embedding of the Tube, we defined core and periphery networks and computed fractal dimension, topological exponent and Rent’s physical exponent for each subnetwork. The core subnetwork has a larger fractal dimension ($d = 0.86$) compared to the peripheral network ($d = 0.61$) when these networks are examined in isolation. However, neither network exhibited topological Rentian scaling. We hypothesize that this is due to the small size of the core network, making it difficult to estimate the topological exponent, and the lack of complexity in the connectivity between the nodes of the periphery.

Despite lack of evidence for topological Rentian scaling, the core network does exhibit physical Rent’s scaling in two dimensions ($p = 0.41$), but not in three. The peripheral network also exhibits Rentian scaling in two dimensions, albeit at a reduced degree ($p = 0.21$), but not in three. Only in the combined network of core and periphery is Rentian scaling present in three dimensions ($p = 0.77$).

4. Discussion
Understanding the topological architecture of the London Tube system offers insight into its evolving role as a mass-transportation system in a large city. Over the last century, the network shifted, enlarged
and adapted with the changing requirements of London’s populace. We investigated the cost efficiency of physical embedding of the present-day Tube and the early Tube network, and we compared these estimates to those expected in benchmark null models. More broadly, the topological and Rentian scaling assessments that we perform here are applicable to other physically embedded systems hypothesized to show a tradeoff between cost and function.

4.1 Topological dimension

Knowing the topological and Euclidean dimensions of the system allows for deeper understanding of the network’s topological space-filling capacity. Although the Euclidean dimension of the system is 2, topologically the system is more similar to a one-dimensional system, with an estimated $d_T$ of 1.12 at small length scales. This is likely due to the fact that the Tube only penetrates a fractal subset of the two-dimensional surface of Greater London. On the whole, the system consists of one-dimensional sections of track with occasional intermingling of pathways, thus leading to a $d_T$ between 1 and 2, and closer to 1. The fact that $d_T > 1$ suggests that the Tube network is more complex than can be represented within a one-dimensional space, consistent with previous estimates [11].

We hypothesize that the difference in topological dimension at varying length scales in the Tube system is a result of the core-periphery architecture of the network. In the core subnetwork, the fractal dimension ($d = 0.86$) is greater than in the periphery ($d = 0.61$), despite significantly fewer nodes present. At small length scales, the topological dimension is averaged over many boxes placed in the relatively large peripheral network (of low topological dimension) with few boxes placed in the smaller core network (of higher topological dimension). However, at larger length scales, fewer large boxes are used to evaluate the network topology, thereby capturing the complexity of the network’s core. Although not explicitly explored in previous studies, many non-homogeneous real networks exhibit this disparity in topology at varying length-scales. Specifically, scaling in brain and VLSI circuits are often assessed using a power law fit to the small and mid-range partition sizes [5], yet the larger box sizes (containing a greater number of nodes) are not included in the fit because their slope is quite different from the small and mid-size partitions. Although this topological shift may result from edge effects, it may also represent an important meso-scale phenomenon that is detectable in topological exponent estimations.

4.2 Topological embedding versus physical embedding

By comparing the topological Rent’s exponent to the physical Rent’s exponent, we conclude that the Tube network is approximately cost-efficiently embedded, because the physical Rent’s exponent is not significantly larger than the topological exponent. However, the stations and connections of the Tube system could be physically embedded more efficiently in order to reach the theoretical minimum (where $p_{\text{min}} = p_T$). This is not unexpected because TFL did not plan the entire Underground system from the beginning.

It is interesting to compare the Tube network to other physically embedded systems. Unlike a VLSI circuit, or even a neurobiological network, the placement of certain nodes in the Tube system is influenced, and therefore constrained, by the needs of neighbourhoods within Greater London. The nodes and track of the Tube were placed progressively over time; therefore, it would be challenging to optimize the network to the degree seen in VLSI circuits (through complex and carefully constructed placement algorithms) [43] or in neuronal networks (through natural selection and plasticity) [44]. The effect of city planning on the cost-efficiency of urban street networks demonstrates that planned, grid-iron networks exhibit a high relative efficiency, with a mid-range level of cost, compared to self-organized, medieval cities that exhibit
rentian scaling for optimal embedding

relatively lower efficiency, but also lower cost [45]. However, the differences in physical embedding found in the Tube network compared to the brain and VLSI circuits may also be attributed to the strict physical constraints of transportation networks, which are often not found in electrical and biological networks. In addition to being near-optimally embedded, VLSIs display Rent’s exponents that are often close to $p = 0.9$ [5], a consequence of the fact that circuit boards display complex topologies in which more edges pass through the boundaries of partitions compared to the number of nodes found within each partition [38]. Interestingly, VLSI circuits placed in series have a similar Rent’s exponent to the Tube system (0.47 and 0.4, respectively) [39]. Unlike parallel circuits, in which wires are placed parallel to each other, series circuits connect nodes along a single path. Similarly to most of the Tube network, series circuits offer a single pathway to a certain destination, whereas parallel circuits have multiple paths to move from point A to point B. The circuits discussed earlier in this paper, with a $p = 0.9$, are typically parallel VLSI circuits. This implies that, although the Tube and circuits in series both exhibit Rentian scaling, they are significantly less complex in design than parallel systems, thus producing a smaller Rent’s exponent.

Importantly, the estimates described above were acquired on the Tube network embedded in two dimensions. We also constructed the network in three dimensions and observed an increase in the Rent’s exponent, implying an increase in the complexity of the network. The increased Rent’s exponent also suggests that it is advantageous to build transportation networks in three dimensions and not restrict stations to a single depth. The ability to change depth allows for multiple tracks to cross one another as well as avoid buildings, streets and sewer systems. Changes in depth become quite dramatic in the periphery of the network, possibly a result of changing land topology from a relatively flat core city to rolling hills on the perimeter of Greater London. However, these elevation changes may also be the product of necessity, such as the building restrictions discussed earlier. Interestingly, in the 1980s the VLSI literature proposed a transition from a two-dimensional circuit network to a three-dimensional one. Although there are inherent challenges to the development of a three-dimensional circuit, studies reported dramatic efficiencies that are not available in two-dimensional circuits [40–42].

Interestingly, the significant increase in Rent’s exponent for the Tube network in two versus three dimensions was not observed in ER random networks or random geometric networks. This finding suggests that the elevation of Tube stations strongly contributes to the cost-efficiency of embedding in physical space, while the same benefit is not passed to randomly wired networks. This benefit may be specifically related to the Tube network structure: a completely connected network that primarily relies on local connections. This is distinct from our benchmark networks that are either notably disconnected (random geometric) or filled with long-range connections (ER). However, it is also important to note that we may not see the Rent’s exponent significantly increase for the model systems, particularly the ER network, because $p$ is already quite high in the 2D network ($p = 0.85$) due to the lack of restrictions on local versus distant connections.

Despite notable changes in physical Rent’s exponent between two- and three-dimensional Tube networks, previous work highlights the fact that transportation networks are fundamentally planar [45]. Our measurement of node versus edge scaling within the spatial Tube network was evaluated using cubes (where $x = y = z$), under the assumption that the network has isotropic tendencies. However, the Tube exhibits significantly different properties along the $x$ and $y$ dimensions (latitude and longitude), which span hundreds or thousands of metres, compared with the $z$ dimension, elevation ATD, with changes in tens of metres. Although discussion of isotropic versus non-isotropic networks is beyond the scope of this paper, it is possible to argue that the Tube network is approximately efficiently embedded into three-dimensional physical space because it is highly restricted along the $z$-axis, thereby restricting the topological Rent’s exponent.
4.3 Minimal wiring hypothesis and cost-efficient embedding

In addition to the Rentian analysis, we performed a second test of cost-efficient embedding by calculating the value $\kappa$ from knowledge of the mean connection distance within the network, the network size, the topological dimension and the embedding dimension. We observed a $\kappa = 0.76$ which is less than that expected in a cost-efficiently embedded system (where $\kappa \sim 1$), but greater than that expected in random benchmark null models ($\kappa = 0.57$). While not optimally wired, the system may be near-optimally wired. Specifically, the Tube network has a topological dimension at small length scales that is smaller than the Euclidean dimension, suggesting that the present Tube network is at a near-minimum wiring state, similar to VLSI and neurobiological systems [5].

4.4 Pre-1900 Tube topophysical properties

A comparison of the modern Tube versus 1900 points to changes within the Tube network and London as a whole over the last 100 years. The lack of clustering and giant hubs in 1900 suggests that many of the Tube lines did not intermingle, and as a result it was likely much more challenging for commuters to quickly change direction. In 1900, TFL had not yet built many of the underground walk-ways between Tube stations that create the small amount of clustering seen in the system today. Additionally, there was no airport at that time, losing yet another clustering feature in the current network. Increased number of hubs and amount of clustering in the later Tube network are likely due to an increase in complexity of the network over time. A more interconnected and complex network translates to more options for commuters and increased ability to quickly reach their final destination. The large interconnect-ability allows fast travel if the shortest path is known. However, recent research suggests that the complexity of many public transportation networks today exceeds the cognitive limits of humans [46].

These changes in topology are accompanied by increases in both topological and Rentian dimensions, driven in part by the intermingling of Tube lines. This change in the fractal embedded nature of the system is consistent with prior hypotheses regarding the development of transportation systems more generally. Specifically, in 2009, Domenech attempted a similar analysis of fractal scaling of transportation networks over time. His work demonstrated that in select underground systems, a transition from small-world topology to fractal scaling topology may be observed during system development [47]. Such a transition may help to explain our observations in the London Tube network over the last century.

4.5 Rentian scaling as a tool to evaluate optimization

Rentian scaling, presented here as a means to evaluate the physical embedding and network optimization of the London Tube network, is a tool that may be applied to any spatial network in which there is a notion of cost which is greater for long edges than for short edges. Through borrowing concepts from electrical engineering, Rentian scaling provides a framework to examine the relative placement of nodes and edges in space and allows us to ask the question: do these components scale together in the physical domain? Networks with greater complexity or ‘logical capacity,’ indicating greater interconnectivity of wiring, naturally exhibit a larger Rent’s exponent ($p$). We argue that for most systems high logical capacity is functionally advantageous because it allows the system to transport objects or data via a variety of paths and configurations. However, with a high level of capacity comes high system costs. To balance capacity with cost, one may consider the topological embedding of the system ($p_T$), which does not rely on physical placement of components.

The combined analysis of physical and topological Rent’s exponent sheds new light on system optimization, as observed differences in Rentian statistics can be driven by differences in topology, differences
in embedding, or both. Early work in spatial networks describes how topological dimensionality of a network often does not reflect the underlying dimensionality of the physical space, yet when it does, it suggests that the network structure is influenced by this space [25]. Although it has been postulated that fractal forms exhibit certain optimal properties, the field of spatial networks does not have well-developed toolboxes to quantify this influence or, going one step further, describe system optimization in the context of physical space. Rentian scaling is capable of connecting the concept of minimal wiring to component placement optimization, thereby offering insights into wiring costs and system efficiency.

4.6 Limitations and future directions

Although this study contributes the first instances of Rentian scaling and three-dimensional architecture in transportation to the network field, there are several limitations that would be of interest to explore in future analyses. One of the most important limitations is that the Rentian analysis focuses on input–output relationships in an effort to assess embedding optimization. However, features of within-box topology are likely also important for understanding the cost of transportation networks. In addition, this analysis uses a simplified view of the Tube with unweighted connections between stations. In the current network, a single connection between two stations often represents two or three track connections between those two stations. Accounting for multiple connections would increase the complexity of the system, likely leading to higher topological and physical Rent’s exponents. In addition, the true complexity of the transportation system could be modelled by treating the Tube, buses and overground trains as a multilayer network [48, 49]. However, it is also possible that the Tube system is not optimized in the Euclidean space, and instead is efficiently embedded in a more abstract one, such as the population density surrounding each station. Further, the inclusion of additional years in the temporal analysis of Tube growth might provide an opportunity to test the hypothesis that the network transitions from a small-world-like topology to a more fractal-like topology. Finally, these questions and the methods presented in this study would benefit from study within the context of additional transport networks.

5. Conclusion

Through examination of a city’s transportation network, it is possible to gain an understanding of the flow of traffic within a city and the places in which people commute most often. In this analysis, we provided insights into features of the Tube network as it functions today and in the year 1900, including the efficiency with which it is embedded into the physical landscape at these time points. Through determination of the topological dimension, physical Rentian scaling and wiring length information, we formed hypotheses surrounding the degree to which the system is physically embedded. Finally, this study offers the first examination of Rentian scaling and physical embedding in a three-dimensional representation of a transportation network.

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